NOTE ON THE THEORY OF THE INTERACTION OF FIELD AND MATTER

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Abstract

The paper develops a method for the systematic integration of the relativistic wave equations for the coupling of electrons and protons with each other and with the electromagnetic field. It is shown that, when the velocity of light is made infinite, these equations reduce to the Schroedinger equation in configuration space for the many body problem. It is further shown that it is impossible on the present theory to eliminate the interaction of a charge with its own field, and that the theory leads to false predictions when it is applied to compute the energy levels and the frequency of the absorption and emission lines of an atom.

THE relativistic theory of the interaction of electrons and protons with each other and with the electromagnetic field has been developed in two papers.[1] The theory is developed in close analogy to the corresponding classical theory: the field is on the one hand determined by the configuration of the charges; and the motion of the charges is affected by the field. The interaction between two charges is not then, on this theory, expressed directly as a function of the configuration of the charges, but as the effect on each of the charges of the field induced by the other. On the classical theory this procedure involves grave difficulties, because each charge reacts also with its own field. The proper energy of this interaction is, for point charges infinite; and it depends upon the motion of the charge. On the classical theory one tried to avoid this difficulty by ascribing to the elementary charges a finite size; but it was not possible to carry through the theory in a way that was not completely arbitrary; nor was it possible to make the work relativistically invariant. One of the purposes of the present paper is to see in how far these difficulties persist in the quantum theory, and in what measure they render impossible the application of the theory.

We may recapitulate briefly the main points of divergence between the present quantum theoretic treatment and the classical theory. In the first place the state of the matter is here represented, not by a trajectory, but by a wave function. Further, the Hamiltonian for the matter is that derived from Dirac's linear wave equation, and not from the quadratic wave equation which would follow from the classical relativistic Hamiltonian. Finally, both the material waves and the electromagnetic waves are quantized, the matter

[1] W. Heisenberg and W. Pauli, Zeits. f. Physik **56**, 1 (1929); ibid. in press. The second of these two papers is referred to in this work as LC. I am greatly indebted to Professor Heisenberg and Professor Pauli, not only for the opportunity of seeing their work before its publication, but also for their very valuable criticism and advice.

to make the particles satisfy the exclusion principle, the field to make the quanta satisfy the Einstein-Bose statistics. This precedure leads to a formal difficulty; for the fourth Maxwell equation

$$\text{div } \boldsymbol{E} - 4\pi\rho = 0 \tag{1}$$

is inconsistent with the quantum conditions, according to which there are functions of the electromagnetic potentials which do not commute with div $\boldsymbol{E}$ but which must commute with the charge density. The two papers of Heisenberg and Pauli are distinguished chiefly by different methods of resolving this difficulty. In the former paper new terms were added to the fourth Maxwell equation to make the new equation consistent with the quantum conditions; in obtaining physical results these terms were to be made to vanish. In the second paper a much more satisfactory method has been used, which takes advantage of the fact that the left hand side of (1) is a constant of the motion for all systems involving matter and radiation; and it is shown that this constancy follows from the gauge invariance of the Hamiltonian for all such systems. The solution of the dynamical problem thus reduces to finding a wave function for the coupled system of field and matter, which makes the Hamiltonian for the coupled system a diagonal matrix, and which in addition makes the left hand side of (1) vanish. The wave function has thus to satisfy not only the Hamiltonian wave equation, but also a series of wave equations which express the fact that (1) is satisfied at all points in space. It is from this set of wave equations that we shall start in this paper; we shall write them first in the form given in LC Eq. (68), in which the wave function is taken as a function of the Cartesian coordinates

$$\boldsymbol{q}_P = (q_P{}^{(1)}, q_P{}^{(2)}, q_P{}^{(3)}) \; ; \; P = 1, 2, \cdots, N$$

and spin variables

$$\sigma_P \; ; \; P = 1, 2, \cdots, N$$

of the N particles in the system, the number of quanta

$$M_{r\lambda}$$

of frequency

$$\nu_r$$

vector of propagation $\boldsymbol{K}_r = (K_r{}^1, K_r{}^2, K_r{}^3)$; $K_r = \nu_r/c$ and polarization λ, and finally a third set of variables P_{r3}, which are essentially the constant components of the electric field parallel to the vectors of propagation $\boldsymbol{K}_r$. We shall first show that it is possible to eliminate the P_{r3}'s from the wave equations in such a way that (1) is automatically satisfied. The reason why this is possible is that the condition (1) determines div $\boldsymbol{E}$ precisely—instead of determining only the relative probabilities of different values of div $\boldsymbol{E}$,—when the configuration of the charges is known, so that the values of the P's so determined can be put at once into the Hamiltonian. When this is done the variables Q_{r3} canonically conjugate to the P's must disappear from the Hamiltonian, since otherwise the Hamiltonian would not be consistent with (1);

we shall show that this is in fact the case, and then proceed to an investigation of the resulting Hamiltonian.

It should be observed that the wave functions must satisfy, in addition to the wave equations in configuration space, the condition that they be antisymmetric in all the electrons of the system, and antisymmetric in all the protons of the system. Only wave functions satisfying these conditions are to be considered in this paper; and it will therefore be unnecessary to indicate the antisymmetry of the wave functions explicitly. The fact that electrons and protons satisfy the exclusion principle is largely irrelevant to the difficulties discussed in this work; for these difficulties persist even in the one-electron problem. On the other hand it is to be hoped that the resolution of the errors in the present theory will make the heuristic postulate of the exclusion principle unnecessary.

We shall look for a solution of the wave equation in which both wave function and characteristic values are expanded in powers of v/c. It has already been shown by Breit[2] that, when the interaction of the particles may be treated as small, and radiation processes may be neglected, that these interaction terms give a contribution to the energy of the system, which, in second order in v/c, agrees with that computed from Breit's equation.[3] We shall not retain the assumption that the interaction terms are small, and shall show, somewhat more generally, that when the proper energy terms are systematically neglected, the characteristic energy levels and the wave function are determined by Breit's equation. This equation is, of course, not relativistically invariant; and it takes no account of radiation processes. In order to remedy these defects one must retain the proper energy terms; and we shall show that it is then possible to make a *formally* satisfactory theory to give the shape and position of all spectral lines, and the energy of the normal state. The theory, is, however, wrong, since it gives a displacement of the spectral lines from the frequency predicted on the basis of the nonrelativistic theory which is in general infinite. This displacement arises from the infinite interaction of the electron with itself; this interaction depends upon the state of the material system; and the difference in the energy for two different states is not in general finite. Thus the present theory gives no more than the nonrelativisitic theory of Jordan, Klein and Wigner.[4] It seems improbable that Breit's equation gives the energy levels of an atom correct to second order terms in v/c; but we shall see that there is ground for supposing that it does give the separation of the fine structure levels correctly in this order. On the other hand the displacement in the frequency of the spectral lines which arises from the proper energy should be of the second order in v/c, and is thus larger than the natural line breadth, which is of the third order; and on the present theory it is not possible to compute this displacement. We shall return later to a consideration of these difficulties.

[2] G. Breit, Phys. Rev. **34**, 553 (1929).

[3] Breit, ibid., Eq. (6).

[4] P. Jordan and O. Klein, Zeits. f. Physik **45**, 751 (1927). P. Jordan and E. Wigner, ibid. **47**, 631 (1928).

1. The condition that the left hand side of (1), regarded as an operator on the wave function ψ of the variables

$$q_P, \sigma_P, M_{r\lambda}, P_{r3}$$

shall make the wave function vanish at every point in space, gives a series of wave equations

$$I \qquad C_r\Psi = \left[P_{r3} + \sum_P e_P v^{r0}(q_P) \right] \psi(q_P, \sigma_P, M_{r\lambda}, P_{r3}) = 0$$

which must hold for all values of the K_r consistent with the boundary conditions. Here the v^{ro} are functions defined in LC (54):

$$v^{r0}(q_P) = \frac{2}{\pi}(ck_r{}^3L^3)^{-1/2} \sin \pi k_r{}^{(1)} q_P{}^{(1)} \sin \pi k_r{}^{(2)} q_P{}^{(2)} \sin \pi k_r{}^{(3)} q_P{}^{(3)} \tag{2}$$

Here L is the length of the fundamental cube, or Hohlraum, and is taken finite in LC to avoid the introduction of a continuous manifold of normal coordinates; in all our results we shall make L become infinite. Furthermore the Hamiltonian for the coupled system, regarded as an operator on the same wave function, gives the wave equation

$$\begin{aligned} II \qquad & \left\{ -E + \sum_r \left[\sum_{\lambda=1,2} M_{r\lambda} h\nu_r + \pi \nu_r P_{r3}{}^2 \right] + \sum_{P=1}^{N} \left[\frac{hc}{2\pi i} (\mathfrak{a}^P \operatorname{grad}_P) + m_P \alpha_0{}^P c^2 \right] \right\} \psi \\ & + \left\{ \sum_{P=1}^{N} \left[e_P A_0{}^0(q_P) + e_P(\mathfrak{a}^P \cdot \boldsymbol{A}^0(q_P)) \right] \right\} \psi \\ & + \left\{ i \sum_r \sum_{\lambda=1,2} \sum_{P=1}^{N} \mu_P{}^{r\lambda} \left[(M_{r\lambda}+1)^{1/2} \Delta_{r\lambda}{}^{-1} - M_{r\lambda}{}^{1/2} \Delta_{r\lambda} \right] \right\} \psi \\ & + \left\{ \sum_r \sum_{P=1}^{N} \mu_P{}^{r3} Q_{r3} \right\} \psi = (+H - E)\psi = 0. \end{aligned}$$

Here $A_0{}^0$ is the external scalar potent al, and $\boldsymbol{A}^0$ the external vector potential; and $\Delta_{r\lambda}$ is an operator which transfoi ms $M_{r\lambda}$ into $M_{r\lambda} - 1$, and leaves all other variables unchanged; the α^p's are operators which operate only on σ_p, and are derived from the Dirac matrices $\alpha_{\rho\sigma}{}^l$ by the definition

$$\alpha_l{}^P F(\sigma_P) = \sum_{\rho P} \alpha_{\sigma_P}{}^l{}_{\rho_P} F(\rho_P) \; ; \quad \alpha_0{}^P F(\sigma_P) = \sum_{\rho P} \alpha_{\sigma_P}{}^0{}_{\rho_P} F(\rho_P).$$

Further the Q_{r3} are canonically conjugate to the P_{r3} so that

$$[P_{r3} Q_{r'3}] = \frac{h}{2\pi i} \delta_{rr'} \tag{3}$$

Finally the functions $\mu_p{}^{r\lambda}$, $\mu_p{}^{r3}$ are defined in LC (59):

$$\mu_P{}^{r3} = e_P c \left(\frac{\nu_r}{2} \right)^{1/2} \sum_{l=1}^{3} \alpha_l{}^P v_l{}^{r3}(q_P)$$

$$\mu_P{}^{r\lambda} = e_P c\left(\frac{h}{2\pi\nu_r}\right)^{1/2} \sum_{l=1}^{3} \alpha_l{}^P v_l{}^{r\lambda}(q_P) \ ; \ \lambda = 1,2 \tag{4}$$

$$v_l{}^{r\lambda}(q_P) = \left(\frac{8}{L^3}\right)^{1/2} F_{l\lambda}^{r} \cos \pi k_l q_P{}^{(l)} \sin \pi k_{l'} q_P{}^{(l')} \sin \pi k_{l''} q_P{}^{(l'')} \ ;$$

$$l = 1,2,3.$$

$$\lambda = 1,2,3.$$

The square matrix $F_{l\lambda}^{r}$ is given by the scheme

l \ $\lambda=$	1	2	3
1	$\epsilon_2{}^r/(\epsilon_1{}^2+\epsilon_2{}^2)^{1/2}$	$\epsilon_1\epsilon_3/(\epsilon_1{}^2+\epsilon_2{}^2)^{1/2}$	ϵ_1
2	$-\epsilon_1/(\epsilon_1{}^2+\epsilon_2{}^2)^{1/2}$	$\epsilon_2\epsilon_3/(\epsilon_1{}^2+\epsilon_2{}^2)^{1/2}$	ϵ_2
3	0	$-(\epsilon_1{}^2+\epsilon_2{}^2)^{1/2}$	ϵ_3

(5)

where the $\epsilon_l{}^r$'s are the direction cosines of the vector $\boldsymbol{K}_r$. It should be observed that

$$\frac{hc}{2\pi i}(\mathfrak{a}^P \ \text{grad}_P) e_P v^{r0}(q_P) = \frac{h}{2\pi i}\mu_P{}^{r3}. \tag{6}$$

This equation, together with (3), shows that

$$[HC_r] = 0 \tag{7}$$

for all r, so that all the C_r's are constants of the motion. The equations I and II are those given[5] in LC (68).

From I we see that the wave function must be singular in the P's. We may avoid the use of singular functions by making a contact transformation from the variables P to Q, and writing the wave function as:

$$\psi(q_P, \sigma_P, M_{r\lambda}, Q_{r3}).$$

For we may then solve (3) by taking

$$P_{r3} = \frac{h}{2\pi i}\frac{\partial}{\partial Q_{r3}}.$$

If now we set

$$\psi(q_P, \sigma_P, M_{r\lambda}, Q_{r3}) = e^{-2\pi i/h \sum_r \sum_p e_p v^{r0}(q_p) Q_{r3}} \phi\,(q_P, \sigma_P, M_{r\lambda}, Q_{r3}) \tag{8}$$

the equations I give us

$$\frac{\partial \phi}{\partial Q_{r3}} = 0 \tag{9}$$

for all r so that

$$\phi = \phi(q_P, \sigma_P, M_{r\lambda}).$$

[5] In LC (68) the A's are dropped.

Further the Hamiltonian II becomes

$$
\begin{aligned}
e^{-2\pi i/h}\sum_r \sum_P e_P v^{r0}(q_P)Q_{r3}\Big\{ &-E+\sum_r\Big[\sum_{\lambda=1,2}M_{r\lambda}h\nu_r+\pi\nu_r\sum_{P,P'}^{N}e_Pe_{P'}v_0{}^{r0}(q_P)v^{r0}(q_{P'})\Big]\\
&+ih\sum_r\sum_P\nu_re_Pv^{r0}(q_P)\frac{\partial}{\partial Q_{r3}}-\frac{h^2}{4\pi}\sum_r\nu_r\frac{\partial^2}{\partial Q_{r3}}+\sum_P\Big[\frac{hc}{2\pi i}(\boldsymbol{\alpha}^P\mathrm{grad}_P)+m_Pc^2\alpha_0{}^P\Big]\\
&-c\sum_r\sum_P e_P(\boldsymbol{\alpha}^P\ \mathrm{grad}_P)v^{r0}(q_P)Q_{r3}+\sum_P e_P[A_0{}^0(q_P)+(\boldsymbol{\alpha}^P\cdot\boldsymbol{A}^0(q_P))] \qquad (10)\\
&+\sum_r\sum_P\mu_P{}^{r3}Q_{r3}+i\sum_r\sum_{\lambda=1,2}\sum_P\mu_P{}^{r\lambda}[(M_{r\lambda}+1)^{1/2}\Delta_{r\lambda}{}^{-1}\\
&-M_{r\lambda}{}^{1/2}\Delta_{r\lambda}]\Big\}\phi(q_P\sigma_PM_r)=0.
\end{aligned}
$$

The terms in Q drop out because of (6); the terms in $\partial/\partial Q$ give nothing because of (9). The equations I and II thus reduce to the single system

$$
\begin{aligned}
&\Big\{-E+H_0+\sum_{r\lambda}M_{r\lambda}h\nu_r+i\sum_{r\lambda P}\mu^{r\lambda}[(M_{r\lambda}+1)^{1/2}\Delta_{r\lambda}{}^{-1}-M_{r\lambda}{}^{1/2}\Delta_{r\lambda}]\Big\}\phi=0\\
&H_0=\sum_P\Big\{\frac{hc}{2\pi i}(\boldsymbol{\alpha}^P\ \mathrm{grad}_P)+m_Pc^2\alpha_0{}^P+\rho_P[A_0{}^0(q_P)+(\boldsymbol{\alpha}^P\cdot\boldsymbol{A}^0(q_P))]\Big]\\
&\qquad\qquad+\pi\sum_{P'}\sum_r e_Pe_{P'}\nu_rv^{r0}(q_P)v^{r0}(q_{P'})\Big\}.
\end{aligned}
\qquad (11)
$$

It is this system which we must now investigate.

The terms

$$G(q_P,q_{P'})=\sum_r\pi\nu_rv^{r0}(q_P)v^{r0}(q_{P'})$$

may readily be evaluated,[6] and give for $L\to\infty$

$$G(q_P,q_{P'})\to-\frac{1}{2r_{PP'}}\ ;\ r_{PP'}=|\,q_P-q_{P'}\,|\ . \qquad (12)$$

The terms for p and p' different give the electrostatic interaction of the two particles; the terms for $p=p'$ give the infinite electrostatic proper energy of the particles; on the present theory it is not possible, as it was on the non-relativistic theory,[4] to eliminate these terms; the physical ground for this impossibility has already been indicated, and lies in the fact that the field acting on any particle is the sum of the fields induced by all particles; it is a consequence of the principle of superposition for the field. These electrostatic proper energy terms do not, however, interfere with the application of the theory, since they are constants, and may be dropped from (11) without altering the form of the wave function. We shall find other infinite proper

[6] W. Heisenberg and W. Pauli, Zeits. f. Physik **56**, 1 (1929). Eq. (115); G. Breit, reference 2 Eq. (57). Breit has independently evaluated the P_{r3} terms in the Hamiltonian; and I am much intebted to him for informing me of his result

energy terms in the course of the work; but these will turn out not to be constants, but to depend upon the configuration of the system; dropping them does alter the form of the wave function.

If we now neglect the coupling between matter and the light quantum field the wave equation reduces to

$$\left[-E+H_0+\sum_{r\lambda} M_{r\lambda}h\nu_r\right]\phi=0 \tag{13}$$

and for the case that no quanta are present we have

$$\left\{-E+\sum_P\left[\frac{hc}{2\pi i}(\boldsymbol{\alpha}^P\ \mathrm{grad}_P)+m_Pc^2\alpha_0{}^P+e_P[A_0{}^0(q_P)+(\boldsymbol{\alpha}^P\cdot\boldsymbol{A}(q_P)]\right]\right.$$
$$\left.-\sum_{PP'}\frac{e_Pe_P{}'}{2r_{PP'}}\right\}\phi(\sigma_P,q_P)=0. \tag{14}$$

We shall show that the terms which we have neglected in (12) are small of the order $(v/c)^2$; and by neglecting other terms of the same order, (13) can be considerably simplified. For consider first the equation

$$\left\{-E+\sum_P\left[\frac{hc}{2\pi i}(\boldsymbol{\alpha}^P\cdot\mathrm{grad}_P)+m_Pc^2\alpha_0{}^P\right]\right\}\phi=0. \tag{15}$$

For N-free uncoupled particles. If we choose all matrices$\| \alpha_{\rho_p}{}^{l}{}_{\sigma_p},\|$ of the form

$$\begin{pmatrix} 0 & 0 & b & a \\ 0 & 0 & c & d \\ \bar{b} & \bar{c} & 0 & 0 \\ \bar{a} & \bar{d} & 0 & 0 \end{pmatrix} \tag{16}$$

and all the $\|\alpha_{\rho_p}{}^{o}{}_{\sigma_p}\|$ of the form:

$$\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{16}$$

and satisfying of course

$$[\alpha^{\mu,P}\alpha^{\nu,P'}]=0 \text{ for } P\neq P' \ ; \ [\alpha^{\mu,P}\alpha^{\nu,P}]^+=2\delta\mu\nu \tag{17}$$

then any $\phi(\sigma_p)$ in which n of the σ_p's have either of the values 3 or 4 will be small compared with any of the ϕ's for which all of the σ_p's have the values 1 or 2 of the order $(v/c)^n$. Now the terms

$$-\sum_{PP}{}'\frac{e_Pe_{P'}}{2r_{PP'}},\quad e_PA_0{}^0(q_P)$$

in (14) do not involve the α^p's, while the terms

$$e_P(\boldsymbol{\alpha}^P\cdot\boldsymbol{A}(q_P))$$

are small of the order v/c; thus as v/c is made to vanish, all the solutions of (14) vanish except those for which all the σ_p's have the values 1 or 2; and (14) reduces to

$$\left[-E+\sum_P\left\{m_Pc^2+e_PA_0{}^0(q_P)-\frac{h^2}{8\pi^2m_P}\Delta_P-\sum_{P'}{}'\frac{e_Pe_{P'}}{2r_{PP'}}\right\}\right]\phi(\sigma_Pq_P)=0 \quad (18)$$

for all σ_p's $=1$ or 2 and

$$\phi(\sigma_Pq_P)=0 \text{ for any } \sigma_P=3 \text{ or } 4$$

which is the Schroedinger equation for the N-body problem.[7] It would thus be possible to take (18) as the starting point for our systematic solution of (11) in powers of v/c. We shall not do this, however, as it would complicate the analysis, and lead to no new results. We shall thus give up the assumption that the α's are written in the form (16), and take for our zero'th approximation to (11), the solutions of (14).*

2. We may write the equations (11) *seriatim*:

$$\sum_{r\lambda}M_{r\lambda}=M=0;(-E+H_0)\phi=-i\sum_{r\lambda P}\mu_P{}^{r\lambda}\phi(1_{r\lambda});\quad \phi=\phi(0_{r\lambda}) \quad (11.1)$$

$$M=1\quad\begin{cases}(-E+h\nu_r+H_0)\phi(1_{r\lambda})=+i\sum_P\mu_P{}^{r\lambda}\phi-\left[i\sum_{r'\lambda'}\sum_P\mu_P{}^{r'\lambda'}\phi(1_{r\lambda},1_{r'\lambda'})\right]\\ \quad -2^{1/2}i\sum_P\mu^{r\lambda}\phi(2_{r\lambda})\end{cases} \quad (11.2)$$

$$M=2\quad\begin{cases}(-E+h(\nu_r+\nu_{r'})+H_0)\phi(1_{r\lambda},1_{r'\lambda'})=+i\sum_P[\mu_P{}^{r\lambda}\phi(1_{r'\lambda'})\\ \quad +\mu_P{}^{r'\lambda'}\phi(1_{r\lambda})]-i\sum_{r''\lambda''}{}'\sum_P\mu_P{}^{r''\lambda''}\phi(1_{r\lambda},1_{r'\lambda'},1_{r''\lambda''})\\ \quad -2^{1/2}i\sum_P[\mu_P{}^{r\lambda}\phi(2_{r\lambda},1_{r'\lambda'})+\mu_P{}^{r'\lambda'}\phi(1_{r\lambda},2_{r'\lambda'})]\end{cases} \quad (11.3)$$

$$\begin{cases}(-E+2h\nu_r+H_0)\phi(2_{r\lambda})=+2^{1/2}i\sum_P\mu_P{}^{r\lambda}\phi(1_{r\lambda})\\ \quad -i\sum_{r'\lambda'}{}'\sum_P\mu_P{}^{r'\lambda'}\phi(2_{r\lambda},1_{r'\lambda'})-3^{1/2}i\sum_P\mu_P{}^{r\lambda}\phi(3_{r\lambda})\end{cases} \quad (11.4)$$

etc.

Now for fixed r, and fixed $\boldsymbol{K}_r$,

$$\mu_P{}^{r\lambda}=0(c^{-1/2})$$

whereas

$$\nu_r=0(c).$$

[7] It is possible to write the two component wave equation when the magnetic interactions are retained up to the order $(v/c)^2$, as has been shown by Breit, reference 2.

* We shall not make explicit use of the fact that the α's are in the form (16); we shall, however, retain the assumption that the α's are small of the order v/c, to obtain $\mu_p{}^{r\lambda}=0(c^{-1/2})$.

Thus we should expect $\phi(1_{r\lambda})$ to be small of the order $c^{-3/2}$, $\phi(1_{r\lambda}, 1_{r'\lambda'})$ to be of the order c^{-3}, and so on, and we should try to find a solution of (11) of the form

$$\begin{aligned} E &= E^{(0)} + E^{(1)} + E^{(2)} \cdots \\ \phi(M_{r\lambda}) &= \phi^{(0)}(M_{r\lambda}) + \phi^{(1)}(M_{r\lambda}) \cdots \end{aligned} \tag{19}$$

with

$$\begin{aligned} E^{(n)} &= 0(c^{-n}) \\ \phi^{(n)}(M_{r\lambda}) &= 0(c^{-n-3/2M}) . \end{aligned}$$

It should be observed that in general there will be certain frequencies for which $\phi(1_{r\lambda})$, $\phi(1_{r\lambda}, 1_{r'\lambda'})$ etc. will not converge uniformly for $c \to \infty$; and that for these frequencies the expansions (19) will be illegitimate. The frequencies for which this convergence is non-uniform are those for which

$$+E - \sum_{r\lambda} M_{r\lambda} h\nu_r$$

is a characteristic value of the homogeneous equations

$$(H_0 - \lambda)\phi = 0 . \tag{20}$$

Such frequencies will not occur if (20) has no solutions for $\lambda < E$, i.e. if the material system is in a normal state; but in general the expansions (19) must be modified; we shall return to this modification later, and shall see that it gives a satisfactory theory of the absorption and emission of radiation; but for the present we shall assume that the atom is in a normal state, so that (19) is justified.

On the present theory there is no normal state for the matter, because states of infinite negative energy are possible; one may in fact show that, on the present theory, Dirac jumps to such states from states of positive energy, jumps in which the energy and momentum lost by the matter are taken up by the field, are not only possible, but infinitely probable. But that the theory should predict this is a token of an error in the theory; and since the Dirac jumps do not seem to be *directly* responsible for the difficulties with which we are, in this work, most concerned, we shall for the present neglect them.

We shall first give a complete solution for the case that we drop all proper energy terms, for the case, that is, that in all double sums of the form

$$\sum_{PP'} F(q_P, \sigma_P, q_{P'}, \sigma_{P'})$$

we may set the terms with p equal to p' equal to zero. This solution is not unique beyond terms of the second order in v/c; for in the higher orders it is no longer possible uniquely to separate proper energy and interaction energy. But we may readily obtain a possible solution:

$$\phi(M_{r\lambda}) = \prod_{r\lambda} \left[\frac{-i \sum_P \mu_P{}^{r\lambda}}{h\nu_r} \right]^{M_{r\lambda}} \phi \tag{21}$$

where

$$\left[-E+H_0-\sum_{r\lambda}\sum_{PP'}{}'\frac{\mu_P{}^{r\lambda}\mu_{P'}{}^{r\lambda}}{h\nu_r}\right]\phi=0. \tag{22}$$

For if we put these values, for example, in (11.2) we get

$$\begin{aligned}&+i\frac{\sum_P\mu_P{}^{r\lambda}}{h\nu_r}\left[-E+H_0-\sum_{r'\lambda'}\sum_{PP'}\frac{\mu_P{}^{r'\lambda'}\mu_{P'}{}^{r'\lambda'}}{h\nu_{r'\lambda'}}+h\nu_r\right]\phi\\&\qquad+\frac{i}{h\nu_r}\left[H_0,\sum_P\mu_P{}^{r\lambda}\right]\phi=i\sum_P\mu_P{}^{r\lambda}\phi\,.\end{aligned} \tag{23}$$

Now in

$$\sum_P[H,\mu_P{}^{r\lambda}]=\sum_{PP'}\left[\left(\frac{hc}{2\pi i}(\boldsymbol{\alpha}^P\cdot\operatorname{grad}_P)+m_Pc^2\alpha_0{}^P+e_P(\boldsymbol{\alpha}^P\cdot\boldsymbol{A}(q_P)\right),\mu_{P'}{}^{r\lambda}\right] \tag{24}$$

and in

$$\sum_{PP'}\mu_P{}^{r'\lambda'}\mu_{P'}{}^{r'\lambda'}$$

we may put

$$\sum_{PP'}\to\sum_{PP'}{}'$$

so that

$$\sum_P[H,\mu_P{}^{r\lambda}]=0 \tag{25}$$

and (11.2) is satisfied. In a similar way it may be shown that all the equations (11) are satisfied.

We may evalute the terms

$$\sum_{r\lambda}\frac{\mu_P{}^{r\lambda}\mu_{P'}{}^{r\lambda}}{h\nu_r}=e_Pe_{P'}\frac{c^2}{2\pi}\sum_{ll'}\alpha_l{}^P\alpha_{l'}{}^{P'}\sum_{r\lambda}\frac{v_l{}^{r\lambda}(q_P)v_{l'}{}^{r\lambda}(q_{P'})}{\nu_r{}^2}$$

in (22) by observing that for $l\neq l'$

$$\sum_{r\lambda}\frac{v_l{}^{r\lambda}(q_P)v_{l'}{}^{r\lambda}(q_{P'})}{\nu_r{}^2}=\frac{\partial}{\partial q^l{}_P{}'}\frac{\partial}{\partial q^{l'}_{P'}}F(q_Pq_{P'})\;;\;\Delta_PF=\frac{\pi}{c^2r_{PP'}}$$

and

$$\sum_{r\lambda}\frac{v_l{}^{r\lambda}(q_P)v_l{}^{r\lambda}(q_{P'})}{v_r{}^2}=\frac{-\pi}{2c^2r_{PP'}}-\frac{\partial^2}{\partial q^l{}_P{}^2}F(q_P,q_{P'})$$

so that

$$\sum_{r\lambda}\frac{\mu_P{}^{r\lambda}\mu_{P'}{}^{r\lambda}}{h\nu_r}=\frac{-e_Pe_{P'}}{4}\left\{\frac{(\boldsymbol{\alpha}^P\cdot\boldsymbol{\alpha}^{P'})}{r_{PP'}}+\frac{(\boldsymbol{\alpha}^P\cdot\boldsymbol{r}_{PP'})(\boldsymbol{\alpha}^{P'}\cdot\boldsymbol{r}_{PP'})}{r^3{}_{PP'}}\right\} \tag{27}$$

This gives for ϕ_0

$$\left\{-E+H_0+\frac{1}{4}\sum_{PP'}{}'\frac{e_Pe_{P'}}{r^3{}_{PP'}}[(\boldsymbol{\alpha}^P\cdot\boldsymbol{\alpha}^{P'})r^2{}_{PP'}+(\boldsymbol{\alpha}^P\cdot\boldsymbol{r}_{PP'})(\boldsymbol{\alpha}^{P'}\cdot\boldsymbol{r}_{PP'})]\right\}\phi=0 \tag{28}$$

This is the equation used by Breit.[3] It is patently not relativistically invariant; this means that the proper energy terms are not invariant, and forces us to

retain these terms, at least in part. Furthermore, we have not, in the deduction of (28), used the fact that the atom is in its normal state; in spite of this there is no sign, in the solution, of processes involving the emission or absorption of radiation; for these processes arise from the interaction of the particles with their own field. We have, therefore, to consider the solution of (11) when the proper energy is not neglected; the retention of these terms will preserve the invariance of the theory, and give us an account of radiation processes, but it leads to results in contradiction with experiment; it makes the validity of (28), even to the second order in v/c, doubtful.

3. We can readily find a solution of the form (19) when E^0 and ϕ^0 correspond to a normal state for the matter; but we cannot find this solution in closed form; nor is there an equation in configuration space, corresponding to (28), for ϕ. If we put (19) in (11.2) etc. we get

$$\phi^{(0)}(1_{r\lambda}) = i \sum_m \frac{b_{0m}^{r\lambda} \phi_m^{(0)}}{h\nu_r + E_m^{(0)} - E_0^{(0)}}\ ; \quad \begin{cases} b_{0m}^{r\lambda} = \sum_P \int \cdots \int dq_1^{(1)} \cdots dq_N^{(3)} \\ \left[\sum_{\sigma' \cdots \sigma N} \bar{\phi}_m^{(0)} \mu_P^{r\lambda} \phi_0^{(0)} \right] \\ (H_0 - E_m^{(0)}) \phi_m^{(0)} = 0 \end{cases} \tag{29}$$

and further

$$\begin{aligned} \phi^{(0)}(1_{r\lambda}, 1_{r'\lambda'}) = &- \sum_{mn} \frac{b_{0m}^{r\lambda} b_{mn}^{r'\lambda'} \phi_n^{(0)}}{[h\nu_r + E_m^{(0)} - E_0^{(0)}][h(\nu_r + \nu_{r'}) + E_n^{(0)} - E_0^{(0)}]} \\ &- \sum_{mn} \frac{b_{0m}^{r'\lambda'} b_{mn}^{r\lambda} \phi_n^{(0)}}{[h\nu_{r'} + E_m^{(0)} - E_0^{(0)}][h(\nu_r + \nu_{r'}) + E_n^{(0)} - E_0^{(0)}]} \end{aligned} \tag{30}$$

$$\phi^{(0)}(2_{r\lambda}) = -\ 2^{1/2} \sum_{mn} \frac{b_{0m}^{r\lambda} b_{mn}^{r\lambda}}{[h\nu_r + E_m^{(0)} - E_0^{(0)}][2h\nu_r + E_r^{(0)} - E_0^{(0)}]} \cdot \text{etc.}$$

Further $E(_0{}^1)$ and $\phi(_0{}^1)$ vanish, and

$$E_0^{(2)} = - \sum_{r\lambda} \sum_m \frac{|\, b_{0n}^{r\lambda} \,|^2}{h\nu_r + E_m^{(0)} - E_0^{(0)}}$$

$$\phi_0^{(2)} = \sum_{r\lambda} \sum_m \sum_{n=0} \frac{b_{0m}^{r\lambda} \bar{b}_{mn}^{r\lambda} \phi_n^{(0)}}{[h\nu_r + E_m^{(0)} - E_0^{(0)}][E_n^{(0)} - E_0^{(0)}]} \cdot$$

Moreover $\phi^1(_r1_\lambda)$, $\phi(_0{}^3)$ and $E(_0{}^3)$ vanish, and

$$\phi^{(2)}(1_{r\lambda}) = i \sum_{r'\lambda'} \sum_{mk} \sum_{n \neq 0} \frac{b_{0m}^{r'\lambda'} \bar{b}_{mn}^{r'\lambda'} b_{nk}^{r\lambda} \phi_k^{(0)}}{[h\nu_{r'} + E_m^{(0)} - E_0^{(0)}][h\nu_r + E_k^{(0)} - E_n^{(0)}][E_n^{(0)} - E_0^{(0)}]} \tag{32}$$

$$E_0^{(4)} = - \sum_{r\lambda} \sum_{r'\lambda'} \sum_{mk} \sum_{n \neq 0} \frac{b_{0m}^{r'\lambda'} \bar{b}_{mn}^{r'\lambda'} b_{nk}^{r\lambda} \bar{b}_{k0}^{r\lambda}}{[h\nu_{r'} + E_m^{(0)} - E_0^{(0)}][h\nu_r + E_k^{(0)} - E_n^{(0)}][E_n^{(0)} - E_0^{(0)}]}$$

etc.

The terms

$$-E_0^{(3)}\phi^{(0)}(1_{r\lambda})$$

and

$$-i\sum_P \sum_{\lambda' r'}\mu_P{}^{r'\lambda'}\phi^{(0)}(1_{r\lambda}1_{r'\lambda'}) - 2^{1/2}i\sum_P \mu_P{}^{r\lambda}\phi^{(0)}(2_{r\lambda})$$

in the equation for $\phi^{(3)}(1_{r\lambda})$ cancel, so that

$$\phi^{(3)}(1_{r\lambda})=0$$

and

$$\phi_0{}^{(5)}=0, \quad E_0{}^{(5)}=0.$$

The expansion for $E^{(0)}$ and $\phi^{(0)}$ can be continued, and only terms of even order in v/c appear.

It will be observed that the interaction terms in (31)

$$-\sum_{r\lambda}\sum_m [h\nu_r + E_m{}^{(0)} - E_0{}^{(0)}]^{-1}\left|\int \bar{\phi}_n{}^{(0)}\mu_P{}^{r\lambda}\phi_0{}^{(0)}d\boldsymbol{q}\right|\left|\int \phi_n{}^{(0)}\mu_{P'}{}^{r\lambda}\phi_0{}^{(0)}d\boldsymbol{q}\right| \tag{33}$$

$$\left[\text{we write } \int d\boldsymbol{q} \text{ for } \int\cdots\int dq_1{}^{(1)}\cdots dq_N{}^{(3)}\sum_{\sigma'\cdots\sigma N}\right]$$

differ from those computed for the second order from (28):

$$-\sum_{r\lambda}\sum_m (h\nu_r)^{-1}\left|\int \bar{\phi}_n{}^{(0)}\mu_P{}^{r\lambda}\phi_0{}^{(0)}d\boldsymbol{q}\right|\left|\int \bar{\phi}_n{}^{(0)}\mu_{P'}{}^{r\lambda}\phi_0{}^{(0)}d\boldsymbol{q}\right| \tag{34}$$

It is possible to express

$$\sum_{r\lambda}\frac{\mu_P{}^{r\lambda}\mu_{P'}{}^{r\lambda}}{h\nu_r + E_m{}^{(0)} - E_0{}^{(0)}}$$

in terms of the confluent hypergeometric functions, but the expressions are too complicated to be suitable for calculation.

The proper energy terms

$$-\sum_{r\lambda}\sum_P\sum_m \frac{\left|\int \bar{\phi}_m{}^{(0)}\mu_P{}^{r\lambda}\phi_0{}^{(0)}d\boldsymbol{q}\right|^2}{h\nu_r + E_m{}^{(0)} - E_0{}^{(0)}} = -\sum_{r\lambda P m} T^{r\lambda}_{0m,P}$$

do not exist, although both

$$\sum_{r\lambda} T^{r\lambda}_{0m,P}$$

and

$$\sum_m T^{r\lambda}_{0m,P}$$

converge; for large ν_r,

$$\sum_m T^{r\lambda}_{0m,P} = 0(1)$$

and for large E

$$\sum_{r\lambda} T^{r\lambda}_{0m,P} = 0(1).$$

The energy level of the normal state is thus infinitely displaced by the interaction of the particles with the field; the question which we have now to consider is whether or not the energy differences between two states are displaced by a finite or an infinite amount.

In order to answer this question we must treat the case of energies which do not correspond to a normal state for the matter; and we have to modify (19), and take account of the emission of radiation by the system. And for this purpose it is convenient to make the dimensions L of the Hohlraum infinite, because that makes the physical interpretation of the solution more immediate. For then ν_r becomes continuous and we may normalize the $v_l{}^{r\lambda}(q_p)$ to the intervals $\Delta\nu\Delta\omega$, where $\Delta\omega$ is the element of solid angle of the unit vector ϵ_r. Furthermore we may treat here, to simplify the writing, the case that E corresponds to the first excited state of the atom, so that there is only one energy E_0 lower than E for which (20) is soluble.

We define the energy of the normal state by

$$E_0 = E_0{}^{(0)} + E_0{}^{(2)} + E_0{}^{(4)} \cdots \tag{35}$$

where $E_0{}^{(2)}$ and $E_0{}^{(4)}$ are given by (31) and (32), and we define the corresponding wave function

$$u_0 = \phi_0{}^{(0)} + \phi_0{}^{(2)} + \phi_0{}^{(4)} \cdots \tag{36}$$

where $\phi_0{}^{(2)}$ is given by (31). We can then extend this definition formally to obtain the energy and wave function of excited states:

$$\begin{aligned} E_m &= E_m{}^{(0)} + E_m{}^{(2)} + E_m{}^{(4)} \cdots ; \\ E_m{}^{(2)} &= -\int d\nu_r \int d\omega_r \sum_{\lambda} \sum_{n} \frac{|\, b^{r\lambda}_{mn} \,|^2}{h\nu_r + E_n{}^{(0)} - E_m{}^{(0)}} \\ u_m &= \phi_m{}^{(0)} + \phi_m{}^{(2)} + \phi_m{}^{(4)} \cdots \text{ etc.} \end{aligned} \tag{37}$$

But in the expressions for $E_m{}^{(2)}$ etc, and $\phi_m{}^{(2)}$ etc, the integrals over ν are now improper, and we have to displace the path of integration around the singularities. This is equivalent to replacing

$$\frac{1}{h\nu + E_n{}^{(0)} - E_m{}^{(0)}} \quad \text{for} \quad E_n{}^{(0)} < E_m{}^{(0)}$$

by

$$\frac{1}{h\nu + E_n{}^{(0)} - E_m{}^{(0)}} \pm i\pi\delta(h\nu + E_n{}^{(0)} - E_m{}^{(0)}) \tag{38}$$

and then taking the principal value of the integrals over ν. Then in general all the E_m's except E_0 are complex. We now transform the $\phi(M_{r\lambda}, q_p\sigma_p)$ by the formulae

$$\phi(M_{r\lambda},m)=\sum_{\sigma P}\int\cdots\int dq_1{}^{(1)}\cdots dq_N{}^{(3)}u_m(q_P,\sigma_P)\,\phi(M_{r\lambda},q_P,\sigma_P) \tag{39}$$

and introduce

$$\overset{r\lambda}{\mu}_{mn,P}=\sum_{\sigma P}\int\cdots\int dq_1{}^{(1)}\cdots dq_N{}^{(3)}\bar{u}_m\mu_P{}^{r\lambda}u_n\,. \tag{40}$$

Then if the $\phi(M, m)$ satisfy the equations, which follow from (11):

$$\begin{aligned}(-E+E_m)\phi(0,m)=&-i\int d\nu_r\int d\omega_r\sum_\lambda\sum_P\overset{r\lambda}{\mu}_{0m,P}F_1(\omega_r\lambda)\delta\left(\nu_r+\frac{E_0-E}{n}\right)\\&-\int d\nu_r\int d\omega_r\sum_\lambda\sum_P\overset{r\lambda}{\mu}_{nm,P}\int d\nu_r'\int d\omega_r'\sum_{\lambda'}\sum_{P'}\overset{r'\lambda'}{\mu}_{0n,P'}F_2(\omega_r,\omega_r',\nu_r',\lambda,\lambda')\\&\qquad\delta\left(\nu_r+\nu_r'+\frac{E_0-E}{h}\right)+\cdots\end{aligned} \tag{41}$$

$$\begin{aligned}\phi(1_{r\lambda},1m)=&\,i\sum_{Pn}\overset{r\lambda}{\mu}_{nm,P}\phi(0,n)\left\{\frac{1}{-E+h\nu_r+E_m}+\frac{i\pi\delta_{m0}}{h}\delta\left(\nu_r+\frac{E_m-E}{h}\right)\right\}\\&+F_1(\omega_{r\lambda})\delta\left(\nu_r+\frac{E_m-E}{h}\right)-i\sum_P\int d\nu_r'\int d\omega_r'\sum_{\lambda'}\overset{r\lambda}{\mu}_{0m,P}\\&\qquad F_2(\omega_r',\omega_r,\nu_r',\lambda,\lambda')\delta\left(\nu_r+\nu_r'+\frac{E_0-E}{h}\right)\cdots\end{aligned} \tag{42}$$

$$\begin{aligned}\phi(1_{r\lambda},1_{r'\lambda'},m)=&\,i\sum_{Pm}\left\{\overset{r'\lambda'}{\mu}_{nm,P}\phi(1_{r\lambda},n)+\overset{r\lambda}{\mu}_{nm,P}\phi(1_{r'\lambda'},n)\right\}\\&\left\{\frac{1}{-E+h(\nu_r+\nu_{r'})+E_m}\pm\frac{i\pi\delta_{m0}}{h}\delta\left(\nu_r+\nu_{r'}+\frac{E_m-E}{h}\right)\right\}\\&+F_2(\omega_r,\omega_r',\nu_r',\lambda,\lambda')\delta\left(\nu_r+\nu_r'+\frac{E_0-E}{h}\right)\cdots\end{aligned}$$

where now the functions F are completely arbitrary, then the $\phi(M_{r\lambda}, q_p, \sigma_p)$ satisfy (11); for each choice of these F's we can obtain a solution of (11). Now the terms

$$\frac{\overset{r\lambda}{\mu}_{nm,P}\phi(0,n)}{-E+h\nu_r+E_m}\ \text{etc.}$$

give a radiation field which does not extend to infinity; and the terms

$$\frac{\pm i\pi\overset{r\lambda}{\mu}_{nm,P}}{h}\phi(0,n)\delta_{m0}\delta\left(\nu_r+\frac{E_0-E}{h}\right)\text{etc.}$$

represent outgoing[8] electromagnetic waves, so that by the choice of the F's

[8] P. A. M. Dirac, Zeits. f. Physik **44**, 585 (1927).

we may determine the radiation incident upon the system. The simplest case is that in which only quanta of the single frequency

$$\nu=\frac{1}{h}(E-E_0)=\nu_r \tag{43}$$

are incident upon the system, so that all the F's vanish except

$$F_1(\omega,\lambda).$$

For this case

$$\phi(0,m)=+\frac{i\sum_{\lambda}\sum_{P}\int d\omega_r \mu_{0m,P}^{r\lambda}F_1(\omega,\lambda)}{E-E_m}=\frac{iG_m{}^{\nu}}{E-E_m}\cdot \tag{44}$$

Now by hypothesis E is to be chosen that only one E_m-E, that for $m=1$, say, is to be small, so that only

$$\phi(0,1)=\frac{iG_1{}^{\nu}}{E-E_1} \tag{45}$$

is large. The probability of absorption to this state is thus proportional to

$$\frac{|G_1{}^{\nu}|^2}{|E-E_1|^2}=\frac{|G_1{}^{\nu}|^2}{|h\nu+E_0-E_1|^2} \tag{46}$$

so that the shape of the absorption line is given by

$$\frac{\text{const.}}{\left|\nu+\frac{1}{h}(E_0-E_1)\right|^2} \tag{47}$$

since $G^{\nu}{}_r$, varies slowly with ν.

If we evaluate E_0, E_1 to the second order in v/c, and drop the higher terms—and this is equivalent to neglecting transitions in which more than one quantum plays a part—(47) reduces to

$$\frac{\text{const.}}{\left[\nu+\frac{1}{h}(E_0{}^{(0)}-E_1{}^{0})+\frac{1}{h}(E_0{}^{(2)}-\frac{1}{2}E_1{}^{(2)}-\frac{1}{2}\overline{E}_1{}^{(2)})\right]+\frac{1}{4h^2}|E_1{}^{(2)}-\overline{E}_1{}^{(2)}|^2}. \tag{48}$$

The absorption line is thus of the same shape as that predicted on the basis of the correspondence principle, and that found, for this case, by Dirac[8], and the half-breadth of the line is

$$\left|\frac{E_1{}^{(2)}-\overline{E}_1{}^{(2)}}{2h}\right|=\frac{\pi}{h}\int d\omega\sum_{\lambda}|b_{01}^{r\lambda}|^2=\frac{1}{4\pi\tau_1} \tag{49}$$

where τ_1 is the natural life time of the state 1.

The center of the absorption line is displaced to the red from $(1/h)\,(E_1^{(0)} - E_0^{(0)})$ by an amount

$$\frac{1}{h}\left(E_0^{(2)} - \frac{1}{2}E_1^{(2)} - \frac{1}{2}\bar{E}_1^{(2)}\right)$$
$$= \frac{1}{h}\int d\nu \int d\omega \sum_{\lambda} \sum_{n} \left\{ \frac{|\, b_{1n}^{r\lambda}\,|^2}{h\nu + E_n^{(0)} - E_1^{(0)}} - \frac{|\, b_{0n}^{r\lambda}\,|^2}{h\nu + E_n^{(0)} - E_0^{(0)}} \right\}$$
$$\sim \frac{1}{h}\int d\nu \int d\omega \sum_{\lambda} \sum_{n} \sum_{PP'} \left\{ \frac{\mu_{1n,P}^{r\lambda} \cdot \mu_{n1,P'}^{r\lambda}}{h\nu + E_n^{(0)} - E_1^{(0)}} - \frac{\mu_{0n,P}^{r\lambda} \cdot \mu_{n0,P'}^{r\lambda}}{h\nu + E_n^{(0)} - E_0^{(0)}} \right\}. \quad (50)$$

Here the principle values are to be taken for all improper integrals over ν. The terms for $p \neq p'$ are just those to be expected from (33) for the displacement of the energy levels by the magnetic interaction of the particles. The terms

$$\frac{1}{h}\int d\nu \int d\omega \sum_{\lambda} \sum_{n} \left\{ \frac{|\, \mu_{1n,P}^{r\lambda}\,|^2}{h\nu + E_n^{(0)} - E_1^{(0)}} - \frac{|\, \mu_{0n,P}^{r\lambda}\,|^2}{h\nu + E_n^{(0)} - E_0^{(0)}} \right\} \quad (51)$$

must be ascribed to the effect of the interaction of the particle with its own field. They may be compared with the formula obtained[8] for the same effect by Dirac, who finds a displacement

$$\frac{1}{h}\int d\nu \int d\omega \sum_{\lambda} \frac{|\, \mu_{01,P}^{r\lambda}\,|^2}{h\nu + E_0^{(0)} - E_1^{(0)}}. \quad (52)$$

There does not appear to be any justification for this result, because in its derivation terms were neglected that are of the same order as those retained. But it is of interest to observe that the integral in (52) exists, and gives a finite displacement of the line of the second order in v/c. This displacement is thus larger than the natural line breadth, which is of the third order. It can be computed[9] when the u's are known. Thus for the first Lyman doublet of hydrogen we find the same displacement, in this order, for both components; it turns out to be

$$+\frac{ch}{32\pi^2 e^2}\frac{1}{\tau_1} = \frac{ch}{32\pi^2 e^2} A_{10}$$

which is about forty times the line breadth. The fact that this term, and the similar terms in (51), are the same, in second order, for the two components of the doublet, suggests that the formulae (33) in which the proper energy is neglected, will give the atomic fine structure splitting correct to the second order.

If we try to compute the displacement from (51), we find that the integrals over ν diverge logarithmically for high frequencies. One can readily

[9] The calculations of the displacements predicted by the results of Dirac were carried through in collaboration with Harvey Hall; and I am indebted to him for permission to quote them here. One must use the retarded potentials to obtain a convergent integral.

see that this is not the result of the neglect of higher order terms, nor of any of the approximations made in the work. The theory thus leads to the false prediction that spectral lines will be infinitely displaced from the values computed by the Bohr frequency condition. The behavior of the expression (51) calls for some comment. As the formula stands, the integral over ν diverges absolutely; this may be verified by evaluating the terms for a free particle. But the question arises whether it is possible so to rearrange the order of the integration over ν the summation over n, and the two integrations involved in the evaluation of the μ_{mn}'s by (40), that the limit $n\to\infty$, $\nu\to\infty$ exists. This cannot be effected by an interchange of the sum over n and the integral over ν; but there is a procedure which, when the E's in the resonance denominators of (51) are dropped, does give an absolutely convergent result. This procedure was suggested by Heisenberg, who showed that, if we first perform the integration over ν and ω and the summation over λ, then sum over all the states n of the same energy, then sum these up to some large but finite energy E, take the difference of the two terms for the state (0) and the state (1), then perform the two integrations over the configuration space, and finally allow E to become infinite, the limit $E\to\infty$ exists, and (51) tends to zero. But if we try to apply this procedure to (51) when the E terms are not dropped, we get for the leading term the divergent result

$$4e_p^2/hc\cdot(E_1^{(0)}-E_0^{(0)})/h\cdot\int^\infty dx/x \tag{53}$$

Nor is there any method for obtaining an absolutely convergent expression for (51). It should be observed that (53) gives us another justification for using (33) to get the fine structure separations correct to the second order.

One can see quite simply that (47) ought not to give a finite line displacement. For consider two states of a free particle; in one let the particle be at rest; in the other let it have the velocity v. Then if the energy of the particle at rest be E, the energy of the moving particle in proper coordinates moving with the particle will also be E. But we know how this energy transforms under a Lorentz transformation; in the original coordinates it will be

$$E\beta \qquad \beta=\left[1-(v/c)^2\right]^{-1/2}$$

and in the same coordinates the difference in energy of the two states, which gives the line displacement, will be

$$E(\beta-1)$$

But this can only be finite if E is finite which, by (31), it is not.

We have treated these difficulties in some detail, because they show that the present theory will not be applicable to any problem where relativistic effects are important, where, that is, we cannot be guided throughout by the limiting case $c\to\infty$. The theory can thus not be applied to a discussion of the structure of the nuclei. It appears improbable that the difficulties discussed in this work will be soluble without an adequate theory of the masses of electron and proton; nor is it certain that such a theory will be possible on the basis of the special theory of relativity.